5/10/23

MATH4030 Tutorial

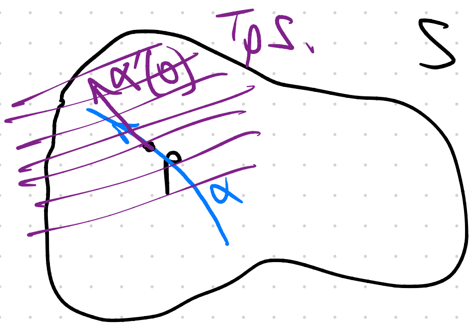
Announcements:

- HW2 due on 16/10.

Recap:

1) Let S be a regular surface. $p \in S$. Then the tangent plane at p is

$$T_pS = \{ \alpha'(0) \in \mathbb{R}^3 : \alpha : (-\varepsilon, \varepsilon) \to S \text{ differentiable}, \ \alpha(0) = p \}.$$



If $X : U \subset \mathbb{R}^2 \to S$, given by $X(u,v)$.

Then $T_pS = \operatorname{span}\{ X_u(p), X_v(p) \}$.

Then $\{X_u, X_v\}$ are lin indep.

means that $\operatorname{span}\{X_u(p), X_v(p)\} \cong \mathbb{R}^2$.

1^{st} f.f. / metric : $g_p : T_pS \times T_pS \to \mathbb{R}$

s.t. $g_p(Z,W) = \langle Z, W\rangle$, for $Z, W \in T_pS$.
standard inner product on $\mathbb{R}^2$.

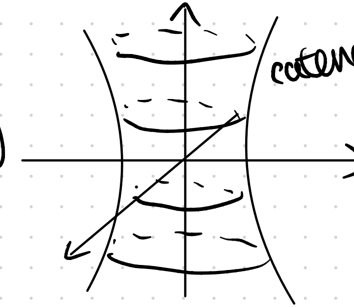
In the basis $\{X_u(p), X_v(p)\}$, $[g] = \begin{bmatrix} g_p(X_u,X_u) & g_p(X_u,X_v) \\ g_p(X_v,X_u) & g_p(X_v,X_v) \end{bmatrix}$

$= \begin{bmatrix} E & F \\ F & G \end{bmatrix}$.

$\det[g] = EG-F^2$.

$$A(R) = \int_R \sqrt{EG-F^2}\, dv\, du.$$

Q1: Param. of the catenoid



$$X(u,v) = \left(c\cosh\left(\tfrac{v}{c}\right)\cos u,\ c\cosh\left(\tfrac{v}{c}\right)\sin u,\ v\right)$$

$$0 < u < 2\pi,\quad v \in \mathbb{R}. \qquad c > 0 \text{ const.}$$

Calculate the 1^{st} f.f. and then compute the surface area for u from 0 to 2π, v from -1 to 1.

Pf: $X_u = \left(-c\cosh\left(\tfrac{v}{c}\right)\sin u,\ c\cosh\left(\tfrac{v}{c}\right)\cos u,\ 0\right)$

$X_v = \left(\sinh\left(\tfrac{v}{c}\right)\cos u,\ \sinh\left(\tfrac{v}{c}\right)\sin u,\ 1\right).$

$$E = \langle X_u, X_u\rangle = c^2\cosh^2\left(\tfrac{v}{c}\right)\sin^2 u + c^2\cosh^2\left(\tfrac{v}{c}\right)\cos^2 u = c^2\cosh^2\left(\tfrac{v}{c}\right).$$

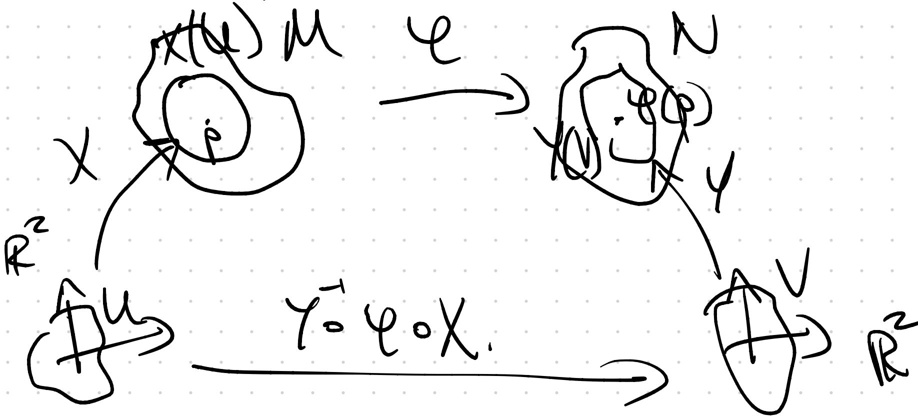
$$F = -c\sinh\left(\tfrac{v}{c}\right)\cosh\left(\tfrac{v}{c}\right)\cos u\sin u + c\cosh\left(\tfrac{v}{c}\right)\sinh\left(\tfrac{v}{c}\right)\cos u\sin u = 0.$$

$$G = \sinh^2\left(\tfrac{v}{c}\right) + 1 = \cosh^2\left(\tfrac{v}{c}\right).$$

$$A(R)=\int_{-1}^{1}\int_{0}^{2\pi}\sqrt{c^2\cosh^4\left(\tfrac{v}{c}\right)}\,du\,dv=\int_{-1}^{1}\int_{0}^{2\pi}c\cosh^2\left(\tfrac{v}{c}\right)du\,dv=\int_{-1}^{1}2\pi c\cosh^2\left(\tfrac{v}{c}\right)dv$$

$$=2\pi c\left(\frac{c}{4}\sinh\left(\frac{2v}{c}\right)\Big|_{-1}^{1}+\frac{v}{2}\Big|_{-1}^{1}\right)=\pi c^2\sinh\left(\frac{2}{c}\right)+2\pi c.$$

Recall: M, N regular surfaces. $\varphi: M\to N$ is differentiable at $p\in M$ if given param. $X: U\subset\mathbb{R}^2\to M$, $Y: V\subset\mathbb{R}^2\to N$ with $p\in X(U)$, $X(U)\subseteq Y(V)$, the map $Y^{-1}\circ\varphi\circ X: U\to V$ is differentiable (as a map $\mathbb{R}^2\to\mathbb{R}^2$).



φ is a diffeomorphism if it is differentiable and $\exists$ a differentiable inverse.

$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$. ← we often define regular surfaces implicitly

Example 3 of Sec 2-3.

Prop: M, N regular surfaces. $M \subset W \subset \mathbb{R}^3$ with W open; $\varphi: W \to \mathbb{R}^3$ is a differentiable map (as a map $\mathbb{R}^3 \to \mathbb{R}^3$) s.t. $\varphi(M) \subset N$.

Then the restriction $\varphi|_M : M \to N$ is a differentiable map between regular surfaces.

Q2: Construct a diffeomorphism between the sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ and the ellipsoid $E = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}$, $a,b,c>0$ const.

Pf: $f: \mathbb{R}^3 \to \mathbb{R}^3$ $f(x,y,z) = (ax, by, cz)$. ← clearly diff. as a map $\mathbb{R}^3 \to \mathbb{R}^3$.

WTS $f(S^2) = E$. : Let $(u,v,w) \in f(S^2)$. Then $\exists (x_0,y_0,z_0) \in S^2$ s.t.

$$f(x_0,y_0,z_0) = (u,v,w) \Rightarrow ax_0 = u,\ by_0 = v,\ cz_0 = w.$$

$$\Rightarrow x_0 = \frac{u}{a}$$

Since $(x_0, y_0, z_0) \in S^2$, $x_0^2 + y_0^2 + z_0^2 = 1$

$$\Rightarrow \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 + \left(\frac{w}{c}\right)^2 = 1. \Rightarrow (u,v,w) \in E.$$

$\Rightarrow f(S^2) \subset E.$

$E \subset f(S^2)$: Similarly check.

f_x, f_y, f_z all exist and are cts. so $f|_{S^2} : S^2 \to E$ is a differentiable map between regular surfaces.

$g : \mathbb{R}^3 \to \mathbb{R}^3$ by $g(x,y,z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$ satisfies:

$$\left.\begin{array}{l} \cdot\ g(E) = S^2 \\ \cdot\ g \text{ is differentiable} \end{array}\right\} \quad g : E \to S^2 \text{ is a differentiable map between regular surfaces.}$$

$\cdot\ f \circ g = \mathrm{Id} = g \circ f.$